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Inelastic Buckling of a Deep Spherical Shell Subject to External Pressure

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This paper is concerned with the investigation of the inelastic buckling of a deep spherical shell subject to a uniformly distributed external pressure. The geometry of the shell is considered to be axisymmetrical while the shell thickness may vary as a function of the polar angle. The edge of the shell is supported elastically. The material of the shell is assumed to satisfy the generalized Ramberg-Osgood stress-strain relations and a power law of steady creep. The analysis is based on Sanders' nonlinear theory of thin shells expressed in an incremental form and Hill's theory of inelastic bifurcation. Computations are carried out by a numerical iterative procedure associated with a finite difference method. Solutions are sought for both the axisymmetrical inelastic buckling and the asymmetrical bifurcation.

I. Introduction

IN recent years, new developments in deep ocean exploration demand further refinement and sophistication in the design of submersibles. An important problem encountered in the design of submerged hulls is the selection of a suitable material with high-strength and low-density properties. In view of the nature of the material available and because of the limitations on structural weight, it is necessary that an accurate stress analysis of the hull and a reliable prediction of the collapse load be established.

When the submerged depth is large, the configuration of the hull is usually spherical. The spherical shell can provide a good structural layout from the viewpoint of stress analysis. Furthermore, it has the advantage of low drag during the motion of the hull. ONR's vehicle ALVIN is a typical example of the spherical hull. For this type of pressure hull, when it is submerged to a large depth, the deformation of the hull may become finite and the state of stress in the hull may reach the inelastic range. In a critical condition, structural failure may occur as a result of inelastic buckling.

Since Shanley first introduced the concept of inelastic bifurcation of a column under increasing axial compressive load,[1]

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substantial progress has been made in the field of inelastic buckling of structures. Research along this line can be found in a comprehensive survey paper prepared by Sewell.[2] The general study of the inelastic stability of solids based on the concept of uniqueness of solution was given by Hill.[3,4] He concludes that Shanley's concept of inelastic buckling under increasing load is a natural sequel to his general theory. In Hill's work, a variational principle is established for finite deformation of inelastic solids based on the existence of a convex plastic potential surface. This variational principle is essentially an extension of the well-known principle of minimum potential energy in finite elasticity. Hill's variational principle incorporated with Rayleigh-Ritz technique or Galerkin technique has been employed for the analysis of inelastic buckling of thin shells.[5-7]

Similar to the case of elastic buckling, the critical condition for inelastic buckling is usually sensitive to the initial imperfections existing in the structural geometry or loading condition. In fact, for many structures, such as columns and complete spherical shells, the inelastic bifurcation occurs only when the geometry of the structure and the loading condition are perfect. With imperfections, bifurcation disappears although the collapse load can still exist. It is found that the magnitude of the collapse load depends strongly on the imperfection of the structure.[8,9] This behavior of imperfection sensitivity indicates that the analytical results derived from the variational principle may depend on the assumed deflection function used in the Rayleigh-Ritz technique. To remedy the deficiency introduced by the Rayleigh-Ritz technique, the problem of inelastic stability of shells can be investigated directly from the governing nonlinear shell equations. Typical examples of this approach are found in Refs. 9 and 10.

In this paper, we shall deal with the problems of axisymmetrical collapse and asymmetrical bifurcation of a deep spherical shell in the inelastic range introduced by a uniform external pressure. The model of spherical shell used in our analysis is chosen to be close to the actual shell configuration of the ALVIN vehicle. The thickness distribution of the shell is regarded as axisymmetrical but variable as a function of the polar angle. The edge of the shell is considered to be elastically supported. The shell material is assumed to be isotropic satisfying Ramberg-Osgood stress-strain relations and a power law of steady creep. Our analysis is based on the Kirchhoff assumption with small strains. Sanders' nonlinear theory for finite deformation of thin shells[11] will be employed in our formulation.

II. Basic Equations

Consider a deep spherical shell with radius of the middle surface R subjected to a uniform external pressure P. Let us introduce surface polar spherical coordinates with polar angle θ and longitude β as shown in Fig. 1. In the following, we shall take θ and β as the parametric coordinates. Thus our lines of curvature are meridional lines and circles perpendicular to the polar axis. The edge of the shell is specified by the circle $\theta = \theta_o$. We shall assume that the thickness distribution of the shell is axisymmetrical, i.e., $h = h(\theta)$.

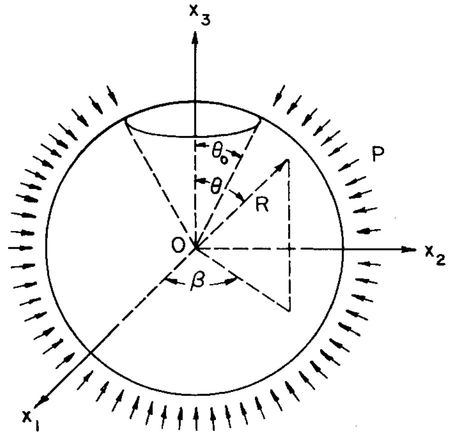

Fig. 1 The geometry of the shell.

Let us denote the components of displacement in the meridional, circumferential, and outward normal directions by U_1, U_2, and W, respectively. According to Sanders' nonlinear theory of shells,[11] the components of rotation in the directions of tangents and normal to the middle surface are given, respectively, by

$$\phi_1 = (1/R)(U_1 - W_{,1}) \tag{1}$$

$$\phi_2 = (1/R)(U_2 - W_{,2}\csc\theta) \tag{2}$$

$$\phi = (1/2R)(U_2\cot\theta + U_{2,1} - U_{1,2}\csc\theta) \tag{3}$$

where $(\ )_{,1} \equiv (\partial/\partial\theta)(\ )$ and $(\ )_{,2} \equiv (\partial/\partial\beta)(\ )$. The membrane strains are

$$E_{11} = (1/R)(W + U_{1,1}) + \tfrac{1}{2}(\phi_1{}^2 + \phi^2) \tag{4}$$

$$E_{22} = (1/R)(W + U_1\cot\theta + U_{2,2}\csc\theta) + \tfrac{1}{2}(\phi_2{}^2 + \phi^2) \tag{5}$$

$$E_{12} = (1/2R)(U_{2,1} + U_{1,2}\csc\theta - U_2\cot\theta + R\phi_1\phi_2) \tag{6}$$

The bending strains are

$$K_{11} = (1/R)\phi_{1,1} \tag{7}$$

$$K_{22} = (1/R)(\phi_{2,2}\csc\theta + \phi_1\cot\theta) \tag{8}$$

$$K_{12} = (1/2R)(\phi_{2,1} + \phi_{1,2}\csc\theta - \phi_2\cot\theta) \tag{9}$$

Let us denote the components of membrane force, transverse shear, and moment resultant by N_{11}, N_{22}, N_{12}, Q_1, Q_2, M_{11}, M_{22} and M_{12}. The equations of equilibrium of the deformed shell are

$$N_{11,1} + N_{11}\cot\theta + N_{12,2}\csc\theta - N_{22}\cot\theta + Q_1 - (\phi_1 N_{11} + \phi_2 N_{12}) - \tfrac{1}{2}[\phi(N_{11} + N_{22})]_{,2}\csc\theta - RP\phi_1 = 0 \tag{10}$$

$$N_{22,2} + 2N_{12}\cos\theta + N_{12,1}\sin\theta + Q_2\sin\theta - (\phi_2 N_{22} + \phi_1 N_{12})\sin\theta + \tfrac{1}{2}[\phi(N_{11} + N_{22})]_{,1}\sin\theta - RP\phi_2\sin\theta = 0 \tag{11}$$

$$Q_{1,1} + Q_1\cot\theta + Q_{2,2}\csc\theta - (N_{11} + N_{22}) - (\phi_1 N_{11} + \phi_2 N_{12})_{,1} - (\phi_1 N_{11} + \phi_2 N_{12})\cot\theta - (\phi_1 N_{12} + \phi_2 N_{22})_{,2}\csc\theta - RP = 0 \tag{12}$$

$$M_{11,1} + M_{11}\cot\theta + M_{12,2}\csc\theta - M_{22}\cot\theta - RQ_1 = 0 \tag{13}$$

$$M_{22,2} + 2M_{12}\cos\theta + M_{12,1}\sin\theta - RQ_2\sin\theta = 0 \tag{14}$$

These equations are self-consistent in a sense that the principle of virtual work can be satisfied. Note that nonlinearity is introduced through the strain-displacement relations and equations of equilibrium.

By the Kirchhoff assumption, the strain components $\bar{E}_{ij}$ $(i = 1, 2;\ j = 1, 2)$ at any point in the shell with a distance z measured from the middle surface can be expressed in terms of the membrane strains and bending strains as

$$\bar{E}_{ij} = E_{ij} + zK_{ij} \tag{15}$$

In Eq. (15), we consider that the shell is sufficiently thin. Hence only the linear term in z is retained. Denote the stress components at any point in the shell by σ_{ij} $(i = 1, 2;\ j = 1, 2)$. In the following we shall assume that the strains are small although the deformation of the shell may be finite. The constitutive relations can be written as

$$\begin{bmatrix} \dot{\bar{E}}_{11} \\ \dot{\bar{E}}_{22} \\ \dot{\bar{E}}_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{12} \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \\ 0 \end{bmatrix} \tag{16}$$

where

$$C_{11} = (1/E) + (G/9)(2\sigma_{11} - \sigma_{22})^2 \tag{17}$$

$$C_{22} = (1/E) + (G/9)(2\sigma_{22} - \sigma_{11})^2 \tag{18}$$

$$C_{33} = (1 + \nu)/E + 2G\sigma_{12}{}^2 \tag{19}$$

$$C_{12} = C_{21} = -(\nu/E) + (G/9)(2\sigma_{11} - \sigma_{22})(2\sigma_{22} - \sigma_{11}) \tag{20}$$

$$C_{13} = C_{31} = (G/3)(2\sigma_{11} - \sigma_{22})\sigma_{12} \tag{21}$$

$$C_{23} = C_{32} = (G/3)(2\sigma_{22}-\sigma_{11})\sigma_{12} \tag{22}$$

$$D_1 = F(\sigma_e)(\sigma_{11}-\tfrac{1}{2}\sigma_{22}) \tag{23}$$

$$D_2 = F(\sigma_e)(\sigma_{22}-\tfrac{1}{2}\sigma_{11}) \tag{24}$$

E is the Young's modulus, ν is the Poisson's ratio, and G is defined as

$$G = \begin{cases} \dfrac{9}{4J_2}\left(\dfrac{1}{E_t}-\dfrac{1}{E}\right) & \text{for } \dot{J}_2 \geqq 0 \text{ and } J_2 = (J_2)_{\max} \\ 0 & \text{for } \dot{J}_2 < 0 \text{ or } J_2 < (J_2)_{\max} \end{cases} \tag{25}$$

E_t is the tangent modulus, J_2 is given as

$$J_2 = \sigma_{11}{}^2+\sigma_{22}{}^2-\sigma_{11}\sigma_{22}+3\sigma_{12}{}^2 \tag{26}$$

$(J_2)_{\max}$ is the maximum value of J_2 in its history, σ_e is the effective stress defined by

$$\sigma_e = J_2{}^{1/2} \tag{27}$$

The function $F(\sigma_e)$ is given by

$$F(\sigma_e) = (1/\sigma_c)(\sigma_e/\sigma_c)^{m-1} \tag{28}$$

where m and σ_c are material constants. The constitutive equation, Eq. (16), is obtained by a generalization of the uniaxial stress-strain relation

$$\dot{\bar{E}} = \begin{cases} \dfrac{\dot{\sigma}}{E_t}+\left(\dfrac{\sigma}{\sigma_c}\right)^m & \text{for } \sigma\dot{\sigma} \geqq 0 \\ \dfrac{\dot{\sigma}}{E}+\left(\dfrac{\sigma}{\sigma_c}\right)^m & \text{for } \sigma\dot{\sigma} < 0 \end{cases} \tag{29}$$

with the assumption that the plastic and creep deformations are incompressible. In our study, we shall assume that the material of the shell satisfies Ramberg-Osgood stress-strain relation for elastic-plastic deformation. Hence the tangent modulus can be expressed as

$$E_t = E[1+\tfrac{3}{7}n_r(\sigma_e/\sigma_o)^{n_r-1}]^{-1} \tag{30}$$

where σ_o is the stress determined by the intercept of the uniaxial stress-strain curve with a straight line through the origin with a slope of $0.7E$ and n_r is a material constant.

When the shell is sufficiently thin, the membrane forces and moments are related to the stress components by

$$N_{ij} = \int_{-h/2}^{h/2} \sigma_{ij}\,dz \tag{31}$$

and

$$M_{ij} = \int_{-h/2}^{h/2} z\sigma_{ij}\,dz \tag{32}$$

Consider h_o as a reference thickness. Let us introduce the following dimensionless quantities:

$$u_i = U_i/h_o,\quad w = W/h_o,\quad \varepsilon_{ij} = (R/h_o)E_{ij},\quad \kappa_{ij} = RK_{ij} \tag{33}$$

$$\lambda = h_o/R,\quad \eta = h/h_o,\quad n_{ij} = N_{ij}/Eh_o,\quad m_{ij} = M_{ij}/Eh_o{}^2 \tag{34}$$

$$q_i = Q_i/Eh_o,\quad p = PR/Eh_o,\quad e_{ij} = (R/h_o)\bar{E}_{ij},\quad \tau_{ij} = \sigma_{ij}/E \tag{35}$$

$$\tau_c = \sigma_c/E,\quad \tau_o = \sigma_o/E,\quad e_t = E_t/E,\quad \zeta = z/h_o \tag{36}$$

Our governing equations for the general deformation of the shell can be expressed in terms of these dimensionless quantities as

$$\phi_1 = \lambda(u_1-w_{,1}) \tag{37}$$

$$\phi_2 = \lambda(u_2-w_{,2}\csc\theta) \tag{38}$$

$$\phi = (\lambda/2)(u_2\cot\theta+u_{2,1}-u_{1,2}\csc\theta) \tag{39}$$

$$e_{11} = w+u_{1,1}+(1/2\lambda)(\phi_1{}^2+\phi^2) \tag{40}$$

$$e_{22} = w+u_1\cot\theta+u_{2,2}\csc\theta+\tfrac{1}{2}\lambda(\phi_2{}^2+\phi^2) \tag{41}$$

$$e_{12} = \tfrac{1}{2}[u_{2,1}+u_{1,2}\csc\theta-u_2\cot\theta+(1/\lambda)\phi_1\phi_2] \tag{42}$$

$$\kappa_{11} = \phi_{1,1} \tag{43}$$

$$\kappa_{22} = \phi_{2,2}\csc\theta+\phi_1\cot\theta \tag{44}$$

$$\kappa_{12} = \tfrac{1}{2}(\phi_{2,1}+\phi_{1,2}\csc\theta-\phi_2\cot\theta) \tag{45}$$

$$n_{11,1}+n_{11}\cot\theta+n_{12,2}\csc\theta- n_{22}\cot\theta+q_1-(\phi_1 n_{11}+\phi_2 n_{12})- \tfrac{1}{2}[\phi(n_{11}+n_{22})]_{,2}\csc\theta-p\phi_1 = 0 \tag{46}$$

$$n_{22,2}+2n_{12}\cos\theta+n_{12,1}\sin\theta+ q_2\sin\theta-(\phi_2 n_{22}+\phi_1 n_{12})\sin\theta+ \tfrac{1}{2}[\phi(n_{11}+n_{22})]_{,1}\sin\theta-p\phi_2\sin\theta = 0 \tag{47}$$

$$q_{1,1}+q_1\cot\theta+q_{2,2}\csc\theta-(n_{11}+n_{22})- (\phi_1 n_{11}+\phi_2 n_{12})_{,1}-(\phi_1 n_{11}+\phi_2 n_{12})\cot\theta- (\phi_1 n_{12}+\phi_2 n_{22})_{,2}\csc\theta-p = 0 \tag{48}$$

$$m_{11,1}+m_{11}\cot\theta+m_{12,2}\csc\theta-m_{22}\cot\theta-(1/\lambda)q_1 = 0 \tag{49}$$

$$m_{22,2}+2m_{12}\cos\theta+m_{12,1}\sin\theta-(1/\lambda)q_2\sin\theta = 0 \tag{50}$$

$$e_{ij} = \varepsilon_{ij}+\zeta\kappa_{ij} \tag{51}$$

$$\begin{bmatrix}\dot{e}_{11}\\ \dot{e}_{22}\\ \dot{e}_{12}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{bmatrix}\begin{bmatrix}\dot{\tau}_{11}\\ \dot{\tau}_{22}\\ \dot{\tau}_{12}\end{bmatrix}+\begin{bmatrix}d_1\\ d_2\\ 0\end{bmatrix} \tag{52}$$

$$c_{11} = (1/\lambda)[1+(g/9)(2\tau_{11}-\tau_{22})^2] \tag{53}$$

$$c_{22} = (1/\lambda)[1+(g/9)(2\tau_{22}-\tau_{11})^2] \tag{54}$$

$$c_{33} = (1/\lambda)(1+\nu+2g\tau_{12}) \tag{55}$$

$$c_{12} = c_{21} = (1/\lambda)[-\nu+(g/9)(2\tau_{11}-\tau_{22})(2\tau_{22}-\tau_{11})] \tag{56}$$

$$c_{13} = c_{31} = (1/\lambda)(g/3)(2\tau_{11}-\tau_{22})\tau_{12} \tag{57}$$

$$c_{23} = c_{32} = (1/\lambda)(g/3)(2\tau_{22}-\tau_{11})\tau_{12} \tag{58}$$

$$d_1 = (1/2\lambda)F(\tau_e)(2\tau_{11}-\tau_{22}) \tag{59}$$

$$d_2 = (1/2\lambda)F(\tau_e)(2\tau_{22}-\tau_{11}) \tag{60}$$

$$j_2 = \tau_{11}{}^2+\tau_{22}{}^2-\tau_{11}\tau_{22}+3\tau_{12}{}^2 \tag{61}$$

$$g = \begin{cases} \dfrac{9}{4j_2}\left(\dfrac{1}{e_t}-1\right) & \text{for } \dot{j}_2 \geqq 0 \text{ and } j_2 = (j_2)_{\max} \\ 0 & \text{for } \dot{j}_2 < 0 \text{ or } j_2 < (j_2)_{\max} \end{cases} \tag{62}$$

$$\tau_e = j_2{}^{1/2} \tag{63}$$

$$F(\tau_e) = (1/\tau_c)(\tau_e/\tau_c)^{m-1} \tag{64}$$

$$e_t = [1+\tfrac{3}{7}n_r(\tau_e/\tau_o)^{n_r-1}]^{-1} \tag{65}$$

$$n_{ij} = \int_{-\eta/2}^{\eta/2} \tau_{ij}\,d\zeta \tag{66}$$

$$m_{ij} = \int_{-\eta/2}^{\eta/2} \tau_{ij}\zeta\,d\zeta \tag{67}$$

The edge of the spherical shell at $\theta = \theta_o$ is considered to be elastically supported. Hence the boundary condition at $\theta = \theta_o$ can be expressed as

$$\begin{bmatrix}\phi_1(\theta_o)\\ u_1(\theta_o)\\ w(\theta_o)\\ u_2(\theta_o)\end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{bmatrix}n_{11}(\theta_o)\\ q_1(\theta_o)\\ m_{11}(\theta_o)\\ m_{12}(\theta_o)\end{bmatrix} \tag{68}$$

where a_{ij} are the elastic constants of the support. At the apex $\theta = \pi$, all displacements and stresses must satisfy regularity conditions. The inelastic deformation of the shell will be determined by the governing equations, Eqs. (37–67), the boundary condition, Eq. (68), and the regularity conditions at $\theta = \pi$.

III. Axisymmetrical Deformation

When the magnitude of the external pressure is small, the deformation of the shell is axisymmetrical with respect to the polar axis. In this case, $\phi_2 = \phi = e_{12} = \kappa_{12} = n_{12} = q_2 = \tau_{12} = 0$ and all physical quantities are independent of β. We shall regard this axisymmetrical deformation as fundamental. In the following, we shall refer the quantities with a super bar to the fundamental deformation of the shell. The governing equations of the fundamental deformation are

$$\bar{\phi}_1 = \lambda(\bar{u}_1-\bar{w}_{,1}) \tag{69}$$

$$\bar{\varepsilon}_{11} = \bar{w}+\bar{u}_{1,1}+(1/2\lambda)\bar{\phi}_1{}^2 \tag{70}$$

$$\bar{\varepsilon}_{22} = \bar{w}+\bar{u}_1\cot\theta \tag{71}$$

$$\bar{\kappa}_{11} = \bar{\phi}_{1,1} \tag{72}$$

$$\bar{\kappa}_{22} = \bar{\phi}_1\cot\theta \tag{73}$$

$$\bar{n}_{11,1}+\bar{n}_{11}\cot\theta-\bar{n}_{22}\cot\theta+\bar{q}_1-\bar{\phi}_1\bar{n}_{11}-p\bar{\phi}_1 = 0 \tag{74}$$

$$\dot{\bar{q}}_{1,1}+\dot{\bar{q}}_1\cot\theta-(\dot{\bar{n}}_{11}+\dot{\bar{n}}_{22})-(\bar{\phi}_1\dot{\bar{n}}_{11})_{,1}-\bar{\phi}_1\dot{\bar{n}}_{11}\cot\theta-\dot{p}=0 \quad (75)$$

$$\dot{\bar{m}}_{11,1}+\dot{\bar{m}}_{11}\cot\theta-\dot{\bar{m}}_{22}\cot\theta-(1/\lambda)\dot{\bar{q}}_1=0 \quad (76)$$

For axisymmetrical deformation, the shearing stress τ_{12} and shearing strain e_{12} vanish. Hence Eq. (52) can be written as

$$\dot{\bar{\mathbf{e}}}=\mathbf{c}\dot{\bar{\boldsymbol{\tau}}}+\mathbf{d} \quad (77)$$

where

$$\bar{\mathbf{e}}=\begin{bmatrix}\bar{e}_{11}\\ \bar{e}_{22}\end{bmatrix},\quad \bar{\boldsymbol{\tau}}=\begin{bmatrix}\bar{\tau}_{11}\\ \bar{\tau}_{22}\end{bmatrix},\quad \mathbf{c}=\begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{bmatrix},\quad \mathbf{d}=\begin{bmatrix}d_1\\ d_2\end{bmatrix} \quad (78)$$

where c_{ij} are calculated by Eqs. (53, 54, and 56) by setting $\tau_{ij}=\bar{\tau}_{ij}$. The inverse relation of Eq. (77) is

$$\dot{\bar{\boldsymbol{\tau}}}=\mathbf{f}\dot{\bar{\mathbf{e}}}-\mathbf{g} \quad (79)$$

where $\mathbf{f}=\mathbf{c}^{-1}$ and $\mathbf{g}=\mathbf{fd}$. Denote

$$\bar{\mathbf{S}}=\begin{bmatrix}\bar{n}_{11}\\ \bar{n}_{22}\\ \bar{m}_{11}\\ \bar{m}_{22}\end{bmatrix},\quad \bar{\boldsymbol{\varepsilon}}=\begin{bmatrix}\bar{\varepsilon}_{11}\\ \bar{\varepsilon}_{22}\\ \bar{\kappa}_{11}\\ \bar{\kappa}_{22}\end{bmatrix} \quad (80)$$

By Eqs. (51, 66, 67, 79, and 80), we have

$$\dot{\bar{\mathbf{S}}}=\mathbf{I}\dot{\bar{\boldsymbol{\varepsilon}}}-\mathbf{J} \quad (81)$$

where $\mathbf{I}$ is a 4×4 matrix and $\mathbf{J}$ is a 4×1 matrix. The elements in $\mathbf{I}$ and $\mathbf{J}$ are

$$I_{11}=\int_{-\eta/2}^{\eta/2} f_{11}\,d\zeta,\quad I_{12}=I_{21}=\int_{-\eta/2}^{\eta/2} f_{12}\,d\zeta$$
$$I_{22}=\int_{-\eta/2}^{\eta/2} f_{22}\,d\zeta,\quad I_{13}=I_{31}=\int_{-\eta/2}^{\eta/2} f_{11}\zeta\,d\zeta$$
$$I_{14}=I_{23}=I_{32}=I_{41}=\int_{-\eta/2}^{\eta/2} f_{12}\zeta\,d\zeta$$
$$I_{24}=I_{42}=\int_{-\eta/2}^{\eta/2} f_{22}\zeta\,d\zeta,\quad I_{33}=\int_{-\eta/2}^{\eta/2} f_{11}\zeta^2\,d\zeta \quad (82)$$
$$I_{34}=I_{43}=\int_{-\eta/2}^{\eta/2} f_{12}\zeta^2\,d\zeta,\quad I_{44}=\int_{-\eta/2}^{\eta/2} f_{22}\zeta^2\,d\zeta$$
$$J_1=\int_{-\eta/2}^{\eta/2} g_1\,d\zeta,\quad J_2=\int_{-\eta/2}^{\eta/2} g_2\,d\zeta$$
$$J_3=\int_{-\eta/2}^{\eta/2} g_1\zeta\,d\zeta,\quad J_4=\int_{-\eta/2}^{\eta/2} g_2\zeta\,d\zeta$$

where f_{ij} and g_i are the elements in $\mathbf{f}$ and $\mathbf{g}$. The inverse relation of Eq. (81) is

$$\dot{\bar{\boldsymbol{\varepsilon}}}=\mathbf{K}\dot{\bar{\mathbf{S}}}+\mathbf{L} \quad (83)$$

where

$$\mathbf{K}=\mathbf{I}^{-1},\quad \mathbf{L}=\mathbf{KJ} \quad (84)$$

By using Eq. (83), the incremental form of the field equations Eqs. (69–76) can be written in the form of the following matrix equations:

$$\dot{\mathbf{U}}'=\mathbf{A}\dot{\mathbf{U}}+\mathbf{B}\dot{\mathbf{V}}+\mathbf{C} \quad (85)$$

and

$$\mathbf{D}\dot{\mathbf{V}}-\mathbf{E}\dot{\mathbf{U}}+\mathbf{F}=\mathbf{0} \quad (86)$$

where the prime represents the differentiation with respect to θ

$$\mathbf{U}=[\bar{\phi}_1\bar{u}_1\bar{w}\bar{n}_{11}\bar{q}_1\bar{m}_{11}]^T \quad\text{and}\quad \mathbf{V}=[\bar{n}_{22}\bar{m}_{22}]^T \quad (87)$$

After eliminating $\dot{\mathbf{V}}$ from Eqs. (85) and (86), we obtain

$$\dot{\mathbf{U}}'=\mathbf{M}\dot{\mathbf{U}}+\mathbf{R} \quad (88)$$

where

$$\mathbf{M}=\mathbf{A}+\mathbf{B}\mathbf{D}^{-1}\mathbf{E} \quad (89)$$

and

$$\mathbf{R}=\mathbf{C}-\mathbf{B}\mathbf{D}^{-1}\mathbf{F} \quad (90)$$

The boundary conditions at $\theta=\theta_o$ can be expressed as

$$\begin{bmatrix}\dot{\bar{\phi}}_1(\theta_o)\\ \dot{\bar{u}}_1(\theta_o)\\ \dot{\bar{w}}(\theta_o)\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}n_{11}(\theta_o)\\ q_1(\theta_o)\\ m_{11}(\theta_o)\end{bmatrix} \quad (91)$$

At the apex, the boundary conditions are

$$\dot{\bar{\phi}}_1(\pi)=\dot{\bar{u}}_1(\pi)=\dot{\bar{q}}_1(\pi)=0 \quad (92)$$

In the following, we shall solve Eqs. (88, 91, and 92) by a modified Euler's method with a finite-difference scheme.

Let n_m be an integer and $\Delta=(\pi-\theta_o)/n_m$, $\theta_i=\theta_o+(i-1)\Delta$, $(i=1,2,\ldots,n_m+1)$. $\mathbf{U}_i=\mathbf{U}(\theta_i)$, $\mathbf{M}_i=\mathbf{M}(\theta_i)$, $\mathbf{R}_i=\mathbf{R}(\theta_i)$. Equation (88) can be expressed as

$$\dot{\mathbf{U}}_{i+1}=\dot{\mathbf{U}}_i+(\Delta/2)(\mathbf{M}_i\dot{\mathbf{U}}_i+\mathbf{M}_{i+1}\dot{\mathbf{U}}_{i+1}+\mathbf{R}_i+\mathbf{R}_{i+1}) \quad (i=1,2,\ldots,n_m) \quad (93)$$

Let $\mathbf{I}_o$ be a unity matrix. Put

$$\mathbf{V}_i=\mathbf{I}_o-(\Delta/2)\mathbf{M}_i \quad (94)$$
$$\mathbf{W}_i=\mathbf{I}_o+(\Delta/2)\mathbf{M}_i \quad (95)$$
$$\mathbf{Z}_i=(\Delta/2)(\mathbf{R}_i+\mathbf{R}_{i+1}) \quad (96)$$
$$\mathbf{P}_i=\mathbf{V}_{i+1}^{-1}\mathbf{W}_i \quad (97)$$

and

$$\mathbf{Q}_i=\mathbf{V}_{i+1}^{-1}\mathbf{Z}_i \quad (98)$$

Equation (93) can be written as

$$\dot{\mathbf{U}}_{i+1}=\mathbf{P}_i\dot{\mathbf{U}}_i+\mathbf{Q}_i \quad (i=1,2,\ldots,n_m) \quad (99)$$

Let

$$\dot{\mathbf{U}}_i=\boldsymbol{\alpha}_i\dot{\mathbf{U}}_1+\boldsymbol{\beta}_i \quad (100)$$

Hence

$$\boldsymbol{\alpha}_1=\mathbf{I}_o \quad\text{and}\quad \boldsymbol{\beta}_1=\mathbf{0} \quad (101)$$

By substitution, we obtain the following recurrence formula for determination of $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$:

$$\boldsymbol{\alpha}_{i+1}=\mathbf{P}_i\boldsymbol{\alpha}_i \quad (102)$$

and

$$\boldsymbol{\beta}_{i+1}=\mathbf{P}_i\boldsymbol{\beta}_i+\mathbf{Q}_i \quad (103)$$

At the apex, we have

$$\dot{\mathbf{U}}_{n_m+1}=\boldsymbol{\alpha}_{n_m+1}\dot{\mathbf{U}}_1+\boldsymbol{\beta}_{n_m+1} \quad (104)$$

Hence, by the boundary conditions, Eqs. (91) and (92), we have

$$\dot{\mathbf{U}}_1=\begin{bmatrix}a_{11}\dot{\bar{n}}_{11}(1)+a_{12}\dot{\bar{q}}_1(1)+a_{13}\dot{\bar{m}}_{11}(1)\\ a_{21}\dot{\bar{n}}_{11}(1)+a_{22}\dot{\bar{q}}_1(1)+a_{23}\dot{\bar{m}}_{11}(1)\\ a_{31}\dot{\bar{n}}_{11}(1)+a_{32}\dot{\bar{q}}_1(1)+a_{33}\dot{\bar{m}}_{11}(1)\\ \dot{\bar{n}}_{11}(1)\\ \dot{\bar{q}}_1(1)\\ \dot{\bar{m}}_{11}(1)\end{bmatrix},\quad \dot{\mathbf{U}}_{n_m+1}=\begin{bmatrix}0\\ 0\\ \dot{\bar{w}}(n_m+1)\\ \dot{\bar{n}}_{11}(n_m+1)\\ 0\\ \dot{\bar{m}}_{11}(n_m+1)\end{bmatrix} \quad (105)$$

The first, second, and fifth equations in Eq. (104) are the simultaneous equations for determination of $\dot{\bar{n}}_{11}(1)$, $\dot{\bar{q}}_1(1)$, and $\dot{\bar{m}}_{11}(1)$. After $\dot{\mathbf{U}}_1$ is calculated, $\dot{\mathbf{U}}_i$ $(i=2,3,\ldots,n_m+1)$ can be determined by Eq. (100). To avoid the singularities involved in the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{E}$ at the apex, we consider that in the neighborhood of the apex, the shell is under a uniform contraction. Hence, at the apex, we have $\dot{\bar{n}}_{11}(\pi)=\dot{\bar{n}}_{22}(\pi)$, $\dot{\bar{m}}_{11}(\pi)=\dot{\bar{m}}_{22}(\pi)$ and $\dot{\bar{\phi}}_1'(\pi)=\dot{\bar{w}}'(\pi)=\dot{\bar{n}}_{11}'(\pi)=\dot{\bar{q}}_1'(\pi)=\dot{\bar{m}}_1'(\pi)=0$. From these conditions, we can find the matrices $\mathbf{M}(\pi)$ and $\mathbf{R}(\pi)$.

Our calculation is based on an iterative procedure, i.e., in each incremental step, we assume a condition of either 1) loading or 2) unloading or reloading and find the increments $\dot{\mathbf{U}}_i$ and $\dot{\mathbf{V}}_i$ for each spatial station. Then, we check the assumed condition. This iterative procedure continues until the calculated condition agrees with the assumed condition at each station of the shell.

It is found that the matrix involved in the evaluation of $\dot{\bar{n}}_{11}(1)$, $\dot{\bar{q}}_1(1)$, and $\dot{\bar{m}}_{11}(1)$ is nearly singular. To improve the accuracy of our computation, the computing program is written in double precision. The inversion of a nearly singular matrix is carried out by an iterative procedure based on the residual derived from the approximate solution in each iterative step.

IV. Asymmetrical Bifurcation

When the external pressure is large, the deformation of the shell may bifurcate from the fundamental axisymmetrical configuration to a shape with an asymmetrical mode. In the

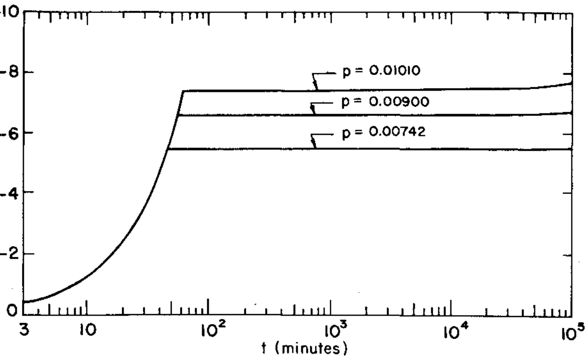

Fig. 2 $\bar{w}_{ave}$ **vs** t **curves.**

following, we shall employ Shanley's concept of inelastic bifurcation under increasing load[1] to analyze the asymmetrical bifurcation of the deep spherical shell. Shanley's concept was later elaborated on by Hill[3,4] based on the argument of uniqueness of solution incorporated with the comparison theorems of linear and nonlinear solids. Hill's conclusion can be stated as follows: during the incremental process of deformation of a solid body, if the constitutive relation is definite and independent of the situations of loading and unloading (or reloading), the relations between stress rates and strain rates are linear. In this case, the material is referred to as a linear solid and the potential function is a quadrate of the strain rate tensor with coefficients dependent on the state of stress. On the other hand, if the constitutive relations are dependent on the situation of loading and unloading, then the form of the potential function would depend on the strain rate tensor and hence the constitutive relations are nonlinear. In this case, the material is referred to as a nonlinear solid. According to these definitions of linear and nonlinear solids it is seen that the potential function W_N for a nonlinear solid would coincide with the quadratic potential function W_L for the corresponding linear solid in a part of the strain rate space. In the region where $W_N \neq W_L$, we have $W_N > W_L$ for most materials. It can be shown by the comparison theorem that if the difference of potentials $W_N - W_L$ is a convex function of the strain rate tensor at every point of the solid body, then the eigenstate in the bifurcation of a nonlinear solid is reached at a higher bifurcation load in comparison with the bifurcation of a corresponding linear solid. Hence, the constitutive relations used for the analysis of bifurcation with the smallest critical load must be linear in the sense just described, i.e., the constitutive relations should be definite and independent of the mode of bifurcation. From the computational point of view, the moduli used for the incremental deformation due to bifurcation should be determined first and the bifurcation analysis is simply a check as to whether the bifurcated deformation is possible. Such a process is similar to what we use for the bifurcation analysis of elastic solids. Note that, under an increasing load, it would be proper to use the constitutive relations obtained from the incremental fundamental deformation of the solid body for the analysis of the critical condition of bifurcation.

In the bifurcation analysis, let us denote the additional displacement components introduced by bifurcation by $\mathbf{u}_i$ and $\mathbf{w}$, the rotations by $\boldsymbol{\phi}_i$ and $\boldsymbol{\phi}$, membrane strains by $\mathbf{e}_{ij}$, bending strains by $\boldsymbol{\kappa}_{ij}$, membrane forces by $\mathbf{n}_{ij}$, transverse shears by $\mathbf{q}_i$, moment resultants by $\mathbf{m}_{ij}$, etc. We have

$$\begin{aligned} &u_1 = \bar{u}_1 + \mathbf{u}_1, \quad u_2 = \mathbf{u}_2, \quad w = \bar{w} + \mathbf{w} \\ &\phi_1 = \bar{\phi}_1 + \boldsymbol{\phi}_1, \quad \phi_2 = \boldsymbol{\phi}_2, \quad \phi = \boldsymbol{\phi} \\ &\varepsilon_{11} = \bar{\varepsilon}_{11} + \boldsymbol{\varepsilon}_{11}, \quad \varepsilon_{22} = \bar{\varepsilon}_{22} + \boldsymbol{\varepsilon}_{22}, \quad \varepsilon_{12} = \boldsymbol{\varepsilon}_{12} \\ &\kappa_{11} = \bar{\kappa}_{11} + \boldsymbol{\kappa}_{11}, \quad \kappa_{22} = \bar{\kappa}_{22} + \boldsymbol{\kappa}_{22}, \quad \kappa_{12} = \boldsymbol{\kappa}_{12} \\ &n_{11} = \bar{n}_{11} + \mathbf{n}_{11}, \quad n_{22} = \bar{n}_{22} + \mathbf{n}_{22}, \quad n_{12} = \mathbf{n}_{12} \\ &q_1 = \bar{q}_1 + \mathbf{q}_1, \quad q_2 = \mathbf{q}_2 \\ &m_{11} = \bar{m}_{11} + \mathbf{m}_{11}, \quad m_{22} = \bar{m}_{22} + \mathbf{m}_{22}, \quad m_{12} = \mathbf{m}_{12} \end{aligned} \tag{106}$$

Substituting Eq. (106) into Eqs. (37–50) and neglecting the second-order terms of those quantities due to bifurcation, we obtain the governing linear equations for the eigenstate of bifurcation where the critical load is involved implicitly in the nonlinear elastic prebifurcation deformation. In the following, we shall assume that $\boldsymbol{\phi}$ is small and drop all terms involving $\boldsymbol{\phi}$ in the field equations. To eliminate the β-dependence in the field equations, let

$$\begin{aligned} &\mathbf{u}_1(\theta,\beta) = u_1(\theta)\sin n\beta, \quad \mathbf{u}_2(\theta,\beta) = u_2(\theta)\cos n\beta, \quad \mathbf{w}(\theta,\beta) = w(\theta)\sin n\beta \\ &\boldsymbol{\phi}_1(\theta,\beta) = \phi_1(\theta)\sin n\beta, \quad \boldsymbol{\phi}_2(\theta,\beta) = \phi_2(\theta)\cos n\beta \\ &\mathbf{e}_{11}(\theta,\beta) = e_{11}(\theta)\sin n\beta, \quad \mathbf{e}_{22}(\theta,\beta) = e_{22}(\theta)\sin n\beta, \quad \mathbf{e}_{12}(\theta,\beta) = e_{12}(\theta)\cos n\beta \\ &\boldsymbol{\kappa}_{11}(\theta,\beta) = \kappa_{11}(\theta)\sin n\beta, \quad \boldsymbol{\kappa}_{22}(\theta,\beta) = \kappa_{22}(\theta)\sin n\beta, \quad \boldsymbol{\kappa}_{12}(\theta,\beta) = \kappa_{12}(\theta)\cos n\beta \\ &\mathbf{n}_{11}(\theta,\beta) = n_{11}(\theta)\sin n\beta, \quad \mathbf{n}_{22}(\theta,\beta) = n_{22}(\theta)\sin n\beta, \quad \mathbf{n}_{12}(\theta,\beta) = n_{12}(\theta)\cos n\beta \\ &\mathbf{q}_1(\theta,\beta) = q_1(\theta)\sin n\beta, \quad \mathbf{q}_2(\theta,\beta) = q_2(\theta)\cos n\beta \\ &\mathbf{m}_{11}(\theta,\beta) = m_{11}(\theta)\sin n\beta, \quad \mathbf{m}_{22}(\theta,\beta) = m_{22}(\theta)\sin n\beta, \quad \mathbf{m}_{12}(\theta,\beta) = m_{12}(\theta)\cos n\beta \end{aligned} \tag{107}$$

where n is an integer. The resulting field equations in our eigenvalue problem are

$$\phi_1 = \lambda(u_1 - w_{,1}) \tag{108}$$

$$\phi_2 = \lambda(u_2 - nw\csc\theta) \tag{109}$$

$$e_{11} = w + u_{1,1} + (1/\lambda)\bar{\phi}_1\phi_1 \tag{110}$$

$$e_{22} = w + u_1\cot\theta - nu_2\csc\theta \tag{111}$$

$$e_{12} = \tfrac{1}{2}[u_{2,1} + nu_1\csc\theta - u_2\cot\theta + (1/\lambda)\bar{\phi}_1\phi_2] \tag{112}$$

$$\kappa_{11} = \phi_{1,1} \tag{113}$$

$$\kappa_{22} = -n\phi_2\csc\theta + \phi_1\cot\theta \tag{114}$$

$$\kappa_{12} = \tfrac{1}{2}(\phi_{2,1} + n\phi_1\csc\theta - \phi_2\cot\theta) \tag{115}$$

$$n_{11,1} + n_{11}\cot\theta - nn_{12}\csc\theta - n_{22}\cot\theta + q_1 - (\bar{\phi}_1 n_{11} + \bar{n}_{11}\phi_1) - p\phi_1 = 0 \tag{116}$$

$$nn_{22} + 2n_{12}\cot\theta + n_{12,1}\sin\theta + q_2\sin\theta - (\bar{n}_{22}\phi_2 + \bar{\phi}_1 n_{12})\sin\theta - p\phi_2\sin\theta = 0 \tag{117}$$

$$\begin{aligned} &q_{1,1} + q_1\cot\theta - nq_2\csc\theta - (n_{11} + n_{22}) - \\ &(\bar{\phi}_1 n_{11,1} + \bar{n}_{11,1}\phi_1 + \bar{\phi}_{1,1}n_{11} + \bar{n}_{11}\phi_{1,1}) - \\ &(\bar{\phi}_1 n_{11} + \bar{n}_{11}\phi_1)\cot\theta + (\bar{\phi}_1 n_{12} + \bar{n}_{22}\phi_2)n\csc\theta = 0 \end{aligned} \tag{118}$$

$$m_{11,1} + m_{11}\cot\theta - nm_{12}\csc\theta - m_{22}\cot\theta - (1/\lambda)q_1 = 0 \tag{119}$$

$$nm_{22}\csc\theta + 2m_{12}\cot\theta + m_{12,1} - (1/\lambda)q_2 = 0 \tag{120}$$

Since the prebifurcation deformation is axisymmetrical, we can set $\tau_{ij} = \bar{\tau}_{ij}$ for $i, j \neq 1, 2$ and $\tau_{12} = 0$ in Eqs. (55, 57, 58, and 61) when we determine the material constants. Hence, if we denote

$$\mathbf{S} = \begin{bmatrix} n_{11} \\ n_{22} \\ m_{11} \\ m_{22} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \kappa_{11} \\ \kappa_{22} \end{bmatrix} \tag{121}$$

then

$$\boldsymbol{\varepsilon} = \mathbf{KS} \tag{122}$$

where $\mathbf{K}$ is given by Eq. (84). The rest of the constitutive relations are

$$\varepsilon_{12} = \bar{K}_{11}n_{12} \tag{123}$$

and

$$\kappa_{12} = \bar{K}_{22}m_{12} \tag{124}$$

where

$$\bar{K}_{11} = (1+\nu)/\lambda\eta, \qquad \bar{K}_{22} = 12(1+\nu)/\lambda\eta^3 \tag{125}$$

Hence Eqs. (123) and (124) are elastic relations. Let

$$\mathbf{U}^* = [\phi_1 w u_1 u_2 q_1 n_{11} m_{11} m_{12}]^T \tag{126}$$

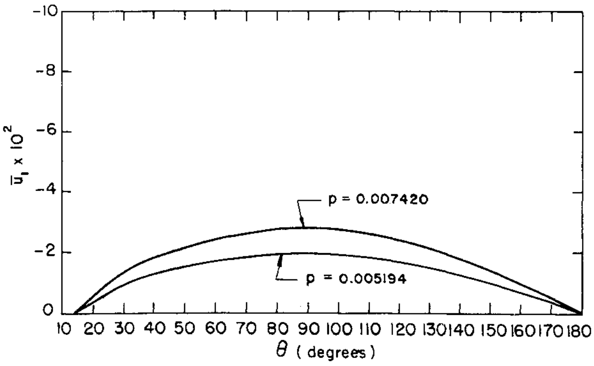

Fig. 3 $\bar{u}_1$ vs θ curves for $p = 0.005194$ and 0.007420.

and

$$\mathbf{V}^* = [\phi_2 q_2 n_{22} m_{22} n_{12}]^T \tag{127}$$

Our governing equations can be written in the form of the following equations:

$$\mathbf{U}^{*\prime} = \mathbf{A}^*\mathbf{U}^* + \mathbf{B}^*\mathbf{V}^* \tag{128}$$

and

$$\mathbf{D}^*\mathbf{V}^* + \mathbf{E}^*\mathbf{U}^* = \mathbf{0} \tag{129}$$

The prebifurcation deformation is involved in the matrices $\mathbf{A}^*$, $\mathbf{B}^*$, $\mathbf{D}^*$, and $\mathbf{E}^*$. After elimination of $\mathbf{V}^*$ from Eqs. (128) and (129), we have

$$\mathbf{U}^{*\prime} = \mathbf{M}^*\mathbf{U}^* \tag{130}$$

where

$$\mathbf{M}^* = \mathbf{A}^* - \mathbf{B}^*\mathbf{D}^{*-1}\mathbf{E}^* \tag{131}$$

Equation (130) can be expressed as the following difference equation:

$$\mathbf{U}^*_{i+1} = \mathbf{P}^*_i\mathbf{U}^*_i \tag{132}$$

where

$$\mathbf{P}^*_i = [\mathbf{I}_o - (\Delta/2)\mathbf{M}^*_{i+1}]^{-1}[\mathbf{I}_o + (\Delta/2)\mathbf{M}^*_i] \tag{133}$$

Thus, if we write

$$\mathbf{U}^*_i = \boldsymbol{\alpha}^*_i\mathbf{U}^*_1 \tag{134}$$

we have

$$\boldsymbol{\alpha}^*_1 = \mathbf{I}_o \tag{135}$$

and

$$\boldsymbol{\alpha}^*_{i+1} = \mathbf{P}^*_i\boldsymbol{\alpha}^*_i \tag{136}$$

Hence all $\boldsymbol{\alpha}_i^*$ $(i = 1, 2, \ldots, n+1)$ can be calculated. Intuitively, we know that when the configuration of the spherical shell is deep enough and the supporting edge is sufficiently rigid, bifurcation would not be governed by the case $n = 1$. The boundary conditions for cases $n \neq 1$ can be written as

$$\mathbf{U}^*_1 = \begin{Bmatrix} a_{11}n_{11}(\theta_o) + a_{12}q_1(\theta_o) + a_{13}m_{11}(\theta_o) + a_{14}m_{12}(\theta_o) \\ a_{31}n_{11}(\theta_o) + a_{32}q_1(\theta_o) + a_{33}m_{11}(\theta_o) + a_{34}m_{12}(\theta_o) \\ a_{21}n_{11}(\theta_o) + a_{22}q_1(\theta_o) + a_{23}m_{11}(\theta_o) + a_{24}m_{12}(\theta_o) \\ a_{41}n_{11}(\theta_o) + a_{42}q_1(\theta_o) + a_{43}m_{11}(\theta_o) + a_{44}m_{12}(\theta_o) \\ q_1(\theta_o) \\ n_{11}(\theta_o) \\ m_{11}(\theta_o) \\ m_{12}(\theta_o) \end{Bmatrix} \tag{137}$$

and

$$\mathbf{U}^*_{n_m+1} = [0\,0\,0\,0\; q_1(\pi) n_{11}(\pi) m_{11}(\pi) m_{12}(\pi)]^T \tag{138}$$

Since

$$\mathbf{U}^*_{n_m+1} = \boldsymbol{\alpha}^*_{n_m+1}\mathbf{U}^*_1 \tag{139}$$

with a substitution of Eqs. (137) and (138) into Eq. (139), a characteristic equation can be obtained by setting the determinant derived from the coefficients of $n_{11}(\theta_o)$, $q_1(\theta_o)$, $m_{11}(\theta_o)$, $m_{12}(\theta_o)$ in the resulting first four equations zero. The critical pressure for bifurcation is then determined by the resulting characteristic equation. In our computation, at each incremental step, we check the value of the characteristic determinant. The critical pressure is determined from the determinant vs pressure curve for each value of n.

V. Results and Discussions

In our numerical computations, we deal with a deep spherical shell whose geometry is similar to the hull of the ALVIN vehicle used for deep ocean exploration. The radius of the middle surface of the shell is 40 in. and the thickness is $h_o = 1.93$ in. The shell contains a hatch with polar angle $\theta_o = 15°$. The actual hull of the ALVIN vehicle also has three viewports. However in our shell model, the openings of viewports are neglected for simplicity of analysis. The shell is thickened in the vicinity of the hatch opening. We assume a linear variation in shell thickness in the region $15° \leqq \theta \leqq 24°$. At the edge of the hatch opening, the shell thickness is 3.6 in. Although the computing program is written to include the general case of elastic support at the edge of the shell, in our computation, we only consider a special case of built-in edge for which $a_{ij} = 0$ $(i = 1, 4$ and $j = 1, 4)$ in Eq. (68).

The material of the shell is 6A1 2Cb 1Ta–0.8 Mo titanium alloy whose inelastic mechanical behavior can be represented with good accuracy by a Ramberg-Osgood stress-strain relation with $E = 16.5 \times 10^6$ psi, $\nu = 0.3$, $\sigma_o = 17.0 \times 10^4$ psi, and $n_r = 50.8$. In our study, the effect of creep under high external pressure is also considered. Because of the lack of the creep data for titanium alloy under room temperature condition, we set $\sigma_c = 65.4 \times 10^4$ psi and $m = 9.9$ in the power law of steady creep in our calculation. A computing program is written in FORTRAN language for the IBM 370 computer. The time interval used in our computation is so chosen that the difference in deflection of the shell is insignificant when we halve the time interval.

The shell is submerged into the sea water at a rate of 285 ft/min (123.6 psi/min) and then remains thereafter at a depth of 13,200 ft where the corresponding external pressure acting on the middle surface of the shell is $p = 0.00742$. When the magnitude of the external pressure is sufficiently small, the deformation of the shell is axisymmetrical. The average normal displacement, $\bar{w}_{ave}$ is plotted against the time in Fig. 2 where the logarithmic scale is used for time. It is found that $\bar{w}_{ave}$ increases as a linear function of time for $p < 0.00742$ and then remains at a constant value when $p = 0.00742$. For this pressure history, the effect of creep is not noticeable for $t < 10^5$ min.

In Figs. 3 and 4, the tangential component of displacement in the meridional direction $\bar{u}_i$ and the normal outward component of displacement $\bar{w}$ are plotted as functions of θ for $p = 0.005194$ and $p = 0.007420$. It is found that in the region distant from the edge, the deformation of the shell is essentially the combination of a uniform contraction and a rigid body translation in the direction of the polar axis. Hence, near the apex, $\bar{u}_1$ and $\bar{w}$ can be expressed approximately as

$$\bar{u}_1 = \gamma \sin\theta, \qquad \bar{w} = \delta - \gamma\cos\theta \tag{140}$$

where δ and γ are constants.

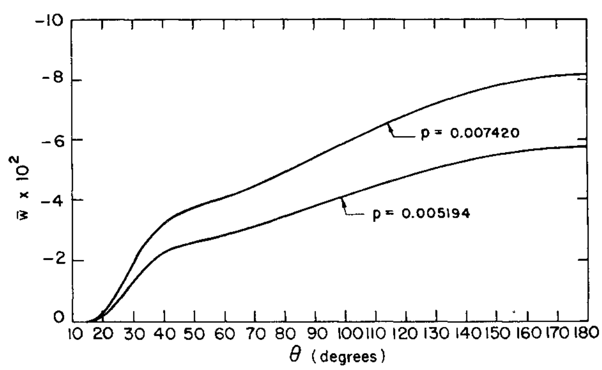

Fig. 4 $\bar{w}$ vs θ curves for $p = 0.005194$ and 0.007420.

In order to study the effect of creep, the spherical hull is submerged to the depth 16,000 ft and 18,000 ft below sea level. The corresponding pressures are $p = 0.00900$ and 0.01010. For these pressures, the $\bar{w}_{ave}$ vs t curves are shown in Fig. 2. It is found that the increment in the average normal deflection due to the steady creep is small. When $p = 0.01010$, it is nearly 3% for $t = 10^5$ min.

Next, we consider the case without creep in which the pressure can increase indefinitely. The pressure vs average normal displacement curve for axisymmetrical deformation is shown in Fig. 5. At $p = p_c = 0.02070$, equivalent to a depth of submergence of 35,000 ft, the p vs $\bar{w}_{ave}$ curve reaches a maximum point which defines the ultimate load for axisymmetrical collapse of the shell. Note that this ultimate load for axisymmetrical collapse of the shell in the inelastic range is much smaller than the classical elastic buckling load of a complete spherical shell which is $(p)_{\text{classical}} = 2/[3(1-\nu^2)]^{1/2} = 1.225$. The critical load for yielding of a complete spherical shell is $(p)_{\text{yielding}} = 2\tau_o = 0.0206$ which is very close to our p_c. In order to determine whether the shell may bifurcate before the ultimate load is reached, we employ the method described in Sec. 4 for the analysis of asymmetrical bifurcation with increasing load. It is found that asymmetrical bifurcation can occur at $p = p_b = 0.01916$ and $n = 2$. The critical depth of submergence for bifurcation is approximately 33,000 ft. As a result of asymmetrical bifurcation, the actual load-carrying capacity will be smaller than the one found based on the axisymmetrical deformation. In order to determine the actual load-carrying-capacity of the shell, postbifurcation analysis must be considered. Note that the difference between p_c and p_b is small and we could anticipate that the actual load-carrying-capacity in the postbifurcation stage would lie somewhere between these two values.

VI. Conclusions

The following conclusions can be drawn about the buckling of deep spherical shells in the inelastic range.

1) When the magnitude of external pressure is sufficiently small, the deformation of the shell is axisymmetrical. Asymmetrical bifurcation occurs when the external pressure reaches a certain critical value.

2) In the prebifurcation stage, the deformation of the shell in the region distant from the edge of the shell is a uniform contraction superposed on a rigid body translation in the direction of the polar axis. Stress concentration occurs in the vicinity of the edge.

3) Under room temperature condition, the effect of creep is usually insignificant within the limit of time of operation.

4) For the deep spherical shell fabricated of titanium alloy, asymmetrical bifurcation occurs when the inelastic deformation of the shell becomes prominent. The actual load-carrying-capacity of the shell after bifurcation is smaller than the limit load determined by the axisymmetrical deformation of the shell. However, the difference is found to be small.

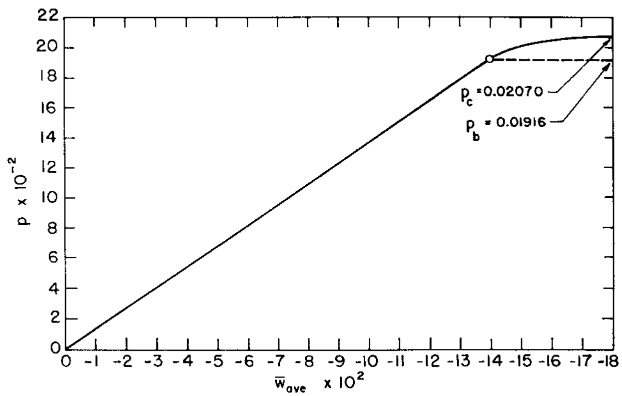

Fig. 5 p **vs** $\bar{w}_{ave}$ **curve.**

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